The multiple tube resonance model

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ABSTRACT

In speech analysis, a recurring acoustical problem is the estimation of resonant structure of a tube of non-uniform cross-sectional area. We model such tubes as a finite sequence of cylindrical tubes of arbitrary, non-uniform length. From this model, we derive a closed form expression of the resonant structure of the model and analytically derive the boundary conditions for the case of a constant group delay. Since it has been noted in the literature that the group delay of the vocal tract is constant, these boundary conditions hold for the vocal tract. In the limiting case, the non-uniform tube model reduces the well studied uniform tube model. For this limiting case, we derive an expression of the tube resonant structure in terms of a Fourier transform. Finally, we derive wave equations from the model, which are consistent with the wave equations for the telegraph wire problem.

Keywords: tube model, tube resonance, transfer function, vocal tract model

1. INTRODUCTION

In speech analysis, a recurring acoustical problem is the estimation of the resonance characteristics of a tube of non-uniform cross-sectional area. The inverse of this problem is the equally classical problem of characterizing the shape of the tube from which a particular sound was emitted. In speech research, the multi-tube model has frequently been used in attempts to relate the shape and configuration of the vocal tract to the observed resonant structure represented by the formants [1]. The motivation in these efforts has been the desire to determine precisely how phonetically distinct sounds are formed and to determine characteristics, such as vocal tract length, which are characteristics of individual speakers [2].

In the modeling tubes, the problem is generally reduced to an equivalent electrical circuit problem. In the continuous case, in which the cross-sectional area is assumed to be a continuous function of the longitudinal position, the problem is precisely equivalent to the telegraph wire problem, in which it is desired to estimate the current out of a (long) wire of non-uniform resistance [3]. In discretizing, the circuit is normally modeled as a connected sequence of lossless delay lines, with loads at connections between the lines. In this discrete (finite) model, the delays are normally modeled as uniform length, resulting in a a Z-transform representation, which is a polynomial in Z. While this representation results in a simple recursive representation of the transfer function [4], it assumes a model based on the partition of the tube into tubes of uniform length. The model does not easily generalize into partitions into arbitrary length tubes, and an heuristic argument is needed to to set the boundary conditions of the problem [1].

We assume the same electrical model common to other approaches (*c.f.* [5, 1, 3, 4]), but we start with the assumption that the given tube is approximated by the concatenation of a finite sequence of tubes, each of which has uniform cross-sectional area, but whose lengths are arbitrary. In so modeling the system, the Z-transform representation does not easily apply. However, a representation may still be obtained, with the advantages that an exact, closed form, representation of the non-uniform length tube model may be obtained; the boundary conditions may be set analytically and the analytically derived boundary conditions agree with the heuristic estimates. In the special case in which each of the tubes in the model have the same length, the closed form representation of the non-uniform tube model is still valid and is seen to simplify to the form of a Fourier transform for the uniform length tube case.

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This paper is structured as follows. In Section 2, we present the model from which the resonance equations are derived. In Section 3, we derive the equations representing the multi-tube resonance presented in Section 2. In that section, we also calculate the boundary conditions for the case of constant group delay and argue that these boundary conditions are valid for the vocal tract. In Section 4, we consider the limiting case, in which all the tubes in the model are of uniform length and demonstrate that the resonance equation for this case reduces to a Fourier transform. Included at the end of the paper are three aPpendices. In Appendix A, we present worked examples of the resonances for the 1, 2 and 4 tube cases. In Appendix B, we present a worked example of the model consisting of N tubes, each of which has the same cross-sectional area. Finally, in Appendix C, we present the derivation of integro-differential equations for the general tube model.

2. A BRIEF NOTE ON THE DERIVATION OF THE TUBE MODEL

The transfer function for the multi-tube model is found by modeling each tube as a forward and backward moving wave with a delay and reflection at each end of the tube. At the interface between each tube, there is a reflection of the right traveling wave to the left, and reflection of the left traveling wave to the right. Within each tube, there is a delay of each of the left and right traveling waves. Therefore, each section of tube is modeled with three junctions:

$L(r)$	The contribution of the left traveling wave with reflection coefficient r
$D(\tau)$	The delay of the right and left traveling waves
$R(r)$	The contribution of the right traveling wave with reflection coefficient r.

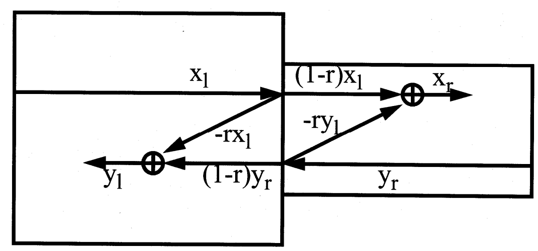

Figure 1. The wave propagation for the general tube model.

The reflection coefficient varies at each tube interface, and is determined from boundary conditions on the wave equation,

$$r_k = \frac{A_{k+1} - A_k}{A_{k+1} + A_k} \quad , k < N \tag{1}$$

where A_k is the area of the kth tube.

For each of the junctions, let $\left[\begin{array}{cc} x_r & y_r \end{array}\right]'$ denote the right and left going wave at the right of the reflection boundary, and let $\left[\begin{array}{cc} x_l & y_l \end{array}\right]'$ denote the wave at the left of the reflection boundary. For the right reflection,

$$\begin{aligned} x_r &= (1+r)\, x_l && (2) \\ y_l &= y_r - r x_l. && (3) \end{aligned}$$

Solving for x_l and y_l, we obtain

$$\left[\begin{array}{c} x_l \\ y_l \end{array}\right] = R(r) \left[\begin{array}{c} x_r \\ y_r \end{array}\right], \tag{4}$$

where

$$R(r) = \left[\begin{array}{cc} \frac{1}{1+r} & 0 \\ \frac{-r}{1+r} & 1 \end{array}\right]. \tag{5}$$

For the left reflection,

$$\begin{aligned} x_r &= x_l + r y_r && (6) \\ y_l &= (1-r)\, y_r. && (7) \end{aligned}$$

Solving for x_l and y_l, we obtain

$$\left[\begin{array}{c} x_l \\ y_l \end{array} \right] = L(r) \left[\begin{array}{c} x_r \\ y_r \end{array} \right], \tag{8}$$

where

$$L(r) = \left[\begin{array}{cc} 1 & -r \\ 0 & 1-r \end{array} \right]. \tag{9}$$

Finally, for the delay we have simply

$$\left[\begin{array}{c} x_l \\ y_l \end{array} \right] = D(r) \left[\begin{array}{c} x_r \\ y_r \end{array} \right] \tag{10}$$

where

$$D(\tau) = \left[\begin{array}{cc} e^{i\omega\tau} & 0 \\ 0 & e^{-i\omega\tau} \end{array} \right]. \tag{11}$$

A complete transfer function for a series of N tubes is the product of these matrices, one for each junction,

$$A = \prod_{k=1}^{N} L(r_{k-1}) D(\tau_k) R(r_k), \tag{12}$$

where r_0 is the reflection at the left end of the first tube, and r_N is the reflection coefficient at the boundary of the final tube. The reflection coefficients, especially at the boundaries, are allowed to be complex to model more general reflection processes, but normally $r_N = 1$, and for a tube closed at the left (i.e. the glottal end) $r_0 = 1$.

The spectral response of the multi-tube system excited by a glottal excitation is found from

$$\frac{1}{V(\omega)} = \left[\begin{array}{cc} 1 & 0 \end{array} \right] A \left[\begin{array}{c} 1 \\ 0 \end{array} \right]. \tag{13}$$

The computed resonance models for the 1, 2 and 4 tube models may be found in Appendix A.

3. THE GENERAL FORM OF THE N-TUBE MODEL

We would now like to determine closed form expressions for the transfer function of the general tube model. We first note that, in general, for each value of k in Eq. (12), the factor $L(r_{k-1}) D(\tau_k) R(r_k)$ introduces a factor of

$$V_k(\omega) = \frac{1 + r_k}{e^{i\omega\tau_k} + r_{k-1} r_k e^{-i\omega\tau_k}} \tag{14}$$

in $V(\omega)$, **where the reflection coefficients r_k in the denominator are idempotent in the sense that $r_k^2 = 1$ for all k.** We may therefore represent $V(\omega)$ as

$$V(\omega) = \prod_{k=1}^{N} V_k(\omega). \tag{15}$$

We may interpret each term Eq. (14) in the product Eq. (15) as contributing two paths to the model. The first path is that in which the signal passes straight through the tube segment, without being reflected and is represented by the expression $e^{i\omega\tau_k}$. In the second path, represented by the expression $r_{k-1} r_k e^{-i\omega\tau_k}$, the signal experiences a reflection at each end of the tube segment. Each of the tube segments is independent in the model, and the denominator of Eq. (15) represents the sum of the signal contributions from each possible path. If we use the notation that a subsystem is the contribution to the denominator of Eq. (15) from signal as it goes through the multiple tubes, with one choice of paths for each tube in the model, the denominator of the N-tube model, Eq. (15), reduces to the sum of all possible subsystems of tubes. If there are N tubes in the model, and each possible combination of tubes forms a subsystem. There are 2^N subsystem, and 2^N terms in the denominator, each representing one of the subsystems.

We will call a connected subset of tubes, within a subsystem, a reflection chain, if the signal within each of the tubes in that subset is reflected. With this notation, the idempotence of the reflection coefficients may be proven by an induction, which we sketch. By Eq. (14), we see that the coefficient of $e^{i\omega(-\tau_1-\tau_2)}$ appears in the denominator

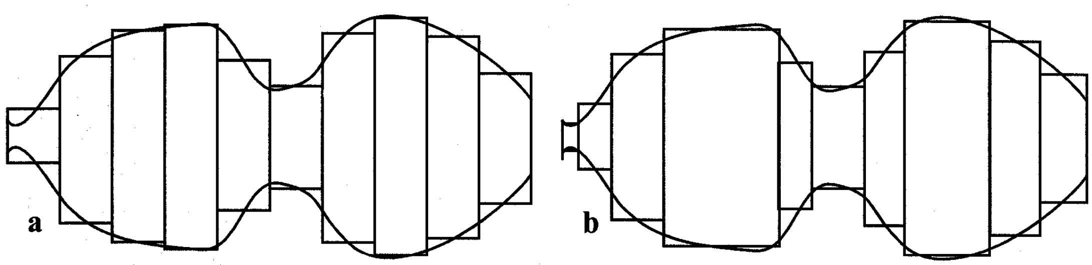

Figure 2. Approximation of a tube of non-uniform cross-section by cylindrical tubes **a**: by uniform length tubes; **b**: by non-uniform length tubes

as r_0r_2, whereas, expanding the denominator as the product of single-tube binomial factors produces a coefficient of $r_0r_1r_1r_2$. For this to be the case, r_1 must be idempotent. Now, assume that the idempotence is valid for an n-tube model. In particular, the reflection coefficients must be idempotent for all reflection chains of length less than n, since all chains of length n or less appear in the n-tube model. The only new case is that in which the entire $n+1$ long tube is a reflection chain. In this case, we can reduce the model to a 2-tube case, in which the first tube is the n-long reflection chain, which acts as one single tube with reflection, and the second tube is the added tube, with reflection. Concatenation of the two tubes, with reflection is therefore reduced to a 2-tube reflection chain.

For reflection chains, the union of all the tubes in that reflection chain may be considered to resonate as a single tube, with reflections at only the two ends of the ensemble of connected tubes. This is simply a re-statement of the idempotence of the reflection coefficients. We may therefore represent each tube in the model as a binary bit in an ordered (finite) binary sequence

$$B(n) = < b_0(n), b_1(n), ..., b_{N-1}(n) > . \tag{16}$$

We may assume the binary ordering of the sequence of subsystems.

$$n = \sum_{k=0}^{N-1} 2^k b_k(n), \tag{17}$$

where $b_k(n) = 1$ if there is a reflection in tube k is in the subsystem s_n and zero otherwise. The index $n \in [0, 2^N - 1]$ provides a complete, ordered, enumeration of the subsystems. We must now characterize the endpoints of each connected connected reflection chain of tubes. We simply note that a binary vector representing the positions of the endpoints may be computed as the exclusive or (XOR) of the binary vector $B(n)$ and a shifted version of the same vector. Specifically, given $B(n)$, we form two new vectors by appending a zero to the beginning and end of $B(n)$, respectively. The new vectors are

$$\begin{aligned} B1(n) &= < 0, b_0(n), b_1(n), \ldots, b_{N-1}(n) > & (18) \\ B2(n) &= < b_0(n), b_1(n), \ldots, b_{N-1}(n), 0 > . & (19) \end{aligned}$$

The vector

$$P(n) = < p_0(n), p_1(n), \ldots, p_{N-1}(n) > = \mathrm{XOR}(B1(n), B2(n)) \tag{20}$$

is the desired vector, with the property that $p_k(n) = 1$ for tube boundaries, which are not interior boundaries of connected tubes in the subsystem, and $p_k(n) = 0$ for tube boundaries which are interior to connected tube segments. It is easily shown that $B(n)$ and $B\left(\left(2^N - 1\right) - n\right)$ are binary compliments of each other. That is

$$B\left(\left(2^N - 1\right) - n\right) = 1 - B(n) \tag{21}$$

In addition, $P(n)$ and $P\left(\left(2^N - 1\right) - n\right)$ have the property that they are identical, except for their first and last elements, which are binary compliments of each other. That is

$$p_k\left(\left(2^N - 1\right) - n\right) = \begin{cases} p_k(n) & , k = 1, ..., N-1 \\ 1 - p_k(n) & .k = 0, N \end{cases} \tag{22}$$

The denominator of the multi-tube transfer function Eq. (15) may be represented as

$$D_{\omega}\left(r_0,...,r_N,\tau_0,...,\tau_{N-1}\right)=\sum_{n=0}^{2^N-1}\left(\prod_{l=0}^{N}r_l^{p_l(n)}\right)\exp\left(i\omega\sum_{l=0}^{N-1}\left(-1\right)^{b_l(n)}\tau_l\right), \tag{23}$$

where the total length T of the vocal tract may be represented as

$$T=\sum_{k=0}^{N-1}\tau_k. \tag{24}$$

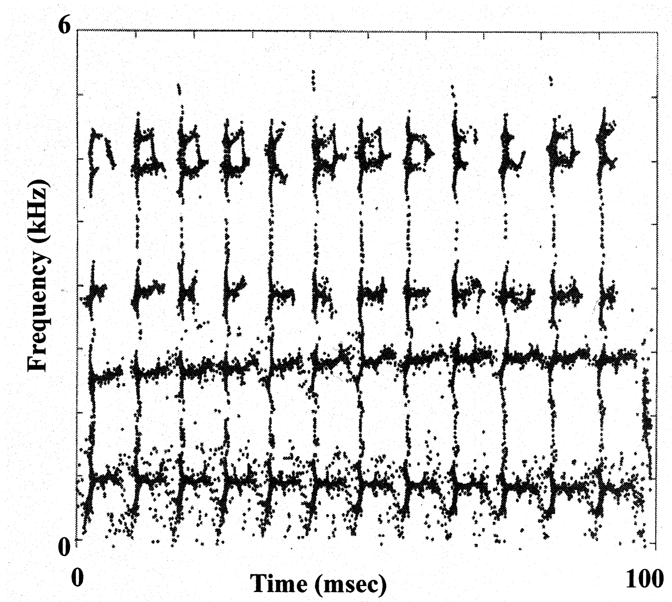

Figure 3. A remapped spectrogram of speech. The nearly vertical lines are the remapped glottal pulses. The fact that all frequencies experience the effects of the excitation pulses simultaneously indicates that the group delay of the vocal tract is nearly constant. For a full discussion, c.f. [9].

So far, we have not related the tube model to any particular physical system. We assume, for the moment, the case of the human vocal tract. It was noted by Nelson that the group delay of uncorrupted speech is constant [9]. For our tube model, this is equivalent to the condition that Eq. (23) is real. We may derive the condition that Eq. (23) is real by setting the imaginary part to zero

$$\Im\{D_{\omega}\left(r_0,...,r_N,\tau_0,...,\tau_{N-1}\right)\}=\sum_{n=0}^{2^N-1}\left(\prod_{l=0}^{N}r_l^{p_l(n)}\right)\sin\left(\omega\sum_{l=0}^{N-1}\left(-1\right)^{b_l(n)}\tau_l\right)\equiv 0, \tag{25}$$

where the identity Eq. (25) must hold for all ω. By the symmetries of Eq. (21) and Eq. (22), Eq. (25) reduces to an expression of the form

$$\Im\{D_{\omega}\left(r_0,...,r_N,\tau_0,...,\tau_{N-1}\right)\}=\sum_{n=0}^{2^{N-1}-1}K_n\left(1-L_n\right)\sin\left(\omega\sum_{l=0}^{N-1}\left(-1\right)^{b_l(n)}\tau_l\right)\equiv 0, \tag{26}$$

where the $K_n\neq 0$ are dependent only on n, and L_n may assume only the values r_0r_N or r_0/r_N. Since, for the case of $n=0$, both r_0 and r_N are introduced into the expression Eq. (25) as factors of the term indexed by 2^N-1,

$$L_0=r_0r_N, \tag{27}$$

regardless of the order N of the tube model. If the values of the τ_k are algebraically independent in the sense that it is not possible to solve for one of them as a linear combination of the others, with rational coefficients, the coefficients $K_n(1-L_n)$ must all be zero. Since the values of K_n can not be zero, $L_n = 1$ for all n. In particular, this must be the case for L_0, with the result that Eq. (26) is true if and only if

$$r_0 = r_N = \pm 1. \tag{28}$$

If the τ_k are not linearly independent, there may be other solutions, but they are not stable, since slight perturbations of the τ_k will result in algebraic independence. Condition Eq. (28) is generally assumed, without proof and is generally not assumed to be related to the group delay of the vocal tract (*c.f.*[1].)

4. THE UNIFORM TUBE MODEL

In the uniform tube model, we assume that the vocal tract is divided into a sequence of tubes, each of which has the same length. The uniform tube model can be extracted from the general model by setting each of the $\tau_k = \tau$ for each value of k. Under this model, Eq. (23) reduces to

$$D_\omega(r_0, ..., r_N, \tau) = \sum_{n=0}^{2^N-1} \left(\prod_{l=0}^{N} r_l^{p_l(n)} \right) \exp\left(i\omega\tau \sum_{l=0}^{N-1} (-1)^{b_l(n)} \right) \tag{29}$$

The exponential in Eq. (29) depends only on the sum of the lengths of the tubes in each particular resonant subsystem, which is represented by the number of bits $b_l(n)$ whose values are unity. If we define the weight W of a binary vector to be the number of 1's in the vector,

$$W(B(n)) = \sum_{k=0}^{N-1} b_k(n) \tag{30}$$

$$W(P(n)) = \sum_{k=0}^{N} p_k(n), \tag{31}$$

and

$$W_k(N) = \{n < 2^N | W(B(n)) = k\}, \tag{32}$$

then

$$\begin{aligned} D_\omega(r_0, ..., r_N, \tau) &= \sum_{n=0}^{2^N-1} \left(\prod_{l=0}^{N} r_l^{p_l(n)} \right) e^{i\omega\tau(N - W(B(n)))} && (33) \\ &= \sum_{L=0}^{N} \sum_{n \in W_L(N)} \left(\prod_{l=0}^{N} r_l^{p_l(n)} \right) e^{i\omega\tau(N-L)} && (34) \\ &= e^{i\omega T} \sum_{L=0}^{N} \left(\sum_{n \in W_L(N)} \prod_{l=0}^{N} r_l^{p_l(n)} \right) e^{-iL\omega\tau}, && (35) \end{aligned}$$

where T is the total delay of the vocal tract.

Eq. (35) expresses the denominator of the transform function in the uniform length tube case as the normal Fourier transform of

$$\sum_{n \in W_L(N)} \prod_{l=0}^{N} r_l^{p_l(n)} \tag{36}$$

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APPENDICIES

APPENDIX A (Computed tube models 1, 2 and 4)

For a single tube model, we let $r_1 \neq 1$ then

$$V(\omega) = \frac{1 + r_1}{e^{i\omega\tau_1} + r_0 r_1 e^{-i\omega\tau_1}}$$

or with $r_0 = r_1 = 1$,

$$V(\omega) = \frac{1}{\cos(\omega\tau)}.$$

For a two-tube model

$$V(\omega) = \frac{(1 + r_1)(1 + r_2)}{e^{i\omega(\tau_1+\tau_2)} + r_0 r_1 e^{i\omega(-\tau_1+\tau_2)} + r_1 r_2 e^{i\omega(\tau_1-\tau_2)} + r_0 r_2 e^{i\omega(-\tau_1-\tau_2)}}$$

and with $r_0 = r_2 = 1$,

$$V(\omega) = \frac{1 + r_1}{r_1 \cos\omega(\tau_1 - \tau_2) + \cos\omega(\tau_1 + \tau_2)}.$$

For the 4-tube model, $V(\omega)$ is $(1 + r_1)(1 + r_2)(1 + r_3)(1 + r_4)$ divided by

$$\begin{array}{rlrlrl}
e^{i\omega(\tau_1+\tau_2+\tau_3+\tau_4)} & + & r_0 r_1 e^{i\omega(-\tau_1+\tau_2+\tau_3+\tau_4)} & + & r_1 r_2 e^{i\omega(\tau_1-\tau_2+\tau_3+\tau_4)} & + \\
r_0 r_2 e^{i\omega(-\tau_1-\tau_2+\tau_3+\tau_4)} & + & r_2 r_3 e^{i\omega(\tau_1+\tau_2-\tau_3+\tau_4)} & + & r_0 r_1 r_2 r_3 e^{i\omega(-\tau_1+\tau_2-\tau_3+\tau_4)} & + \\
r_1 r_3 e^{i\omega(\tau_1-\tau_2-\tau_3+\tau_4)} & + & r_0 r_3 e^{i\omega(-\tau_1-\tau_2-\tau_3+\tau_4)} & + & r_3 r_4 e^{i\omega(\tau_1+\tau_2+\tau_3-\tau_4)} & + \\
r_0 r_1 r_3 r_4 e^{i\omega(-\tau_1+\tau_2+\tau_3-\tau_4)} & + & r_1 r_2 r_3 r_4 e^{i\omega(\tau_1-\tau_2+\tau_3-\tau_4)} & + & r_0 r_2 r_3 r_4 e^{i\omega(-\tau_1-\tau_2+\tau_3-\tau_4)} & + \\
r_2 r_4 e^{i\omega(\tau_1+\tau_2-\tau_3-\tau_4)} & + & r_0 r_1 r_2 r_4 e^{i\omega(-\tau_1+\tau_2-\tau_3-\tau_4)} & + & r_1 r_4 e^{i\omega(\tau_1-\tau_2-\tau_3-\tau_4)} & + \\
r_0 r_4 e^{i\omega(-\tau_1-\tau_2-\tau_3-\tau_4)}, & & & & &
\end{array}$$

or with $r_4 = r_0 = 1$, $V(\omega)$ is $(1 + r_1)(1 + r_2)(1 + r_3)$ divided by

$$\begin{array}{rlrlrl}
r_1 \cos\omega(\tau_1 - \tau_2 - \tau_3 - \tau_4) & + & r_2 \cos\omega(\tau_1 + \tau_2 - \tau_3 - \tau_4) & + & r_1 r_2 r_3 \cos\omega(\tau_1 - \tau_2 + \tau_3 - \tau_4) & + \\
r_3 \cos\omega(\tau_1 + \tau_2 + \tau_3 - \tau_4) & + & r_1 r_3 \cos\omega(\tau_1 - \tau_2 - \tau_3 + \tau_4) & + & r_2 r_3 \cos\omega(\tau_1 + \tau_2 - \tau_3 + \tau_4) & + \\
r_1 r_2 \cos\omega(\tau_1 - \tau_2 + \tau_3 + \tau_4) & + & \cos\omega(\tau_1 + \tau_2 + \tau_3 + \tau_4) & & &
\end{array}$$

APPENDIX B (An example: a partitioned uniform tube)

Consider a tube of length T an constant cross-sectional area. If this tube is broken into N uniform length tubes lf length $\tau = T/N$, the reflection coefficients are $r_k = 0$ for $N > k > 0$. Assuming that

$$0^1 = 0, 0^0 = 1,$$

there are only two non-zero contributions from the expression

$$\sum_{n \in W_k(N)} \prod_{l=0}^{N} r_l^{p_l(n)}.$$

(1) For k=0, there is only one weight zero bi-nomial, for which $p_l(0) \equiv 0$ and

$$\prod_{l=0}^{N} r_l^{p_l(0)} \equiv 1.$$

(2) For $k = N$ there is only one weight N bi-nomial, for which

$$\begin{aligned} p_l(M) &= 1, \quad M = 0, N \\ p_l(M) &= 0, \quad 0 < M < N. \end{aligned}$$

Eq. (35), therefore, reduces to

$$D_\omega(r_0, \ldots, r_N, \tau) = e^{iN\omega\tau} + r_0 r_N e^{-N\omega\tau} = e^{i\omega T} + r_0 r_N e^{-\omega T},$$

which agrees with the single tube model of Appendix A, which was derived directly from the model.

APPENDIX C (Derivation of an integro-differential equation)

Suppose a full tube has length T (distance is measured in time in units of the speed of sound), with cross sectional area function $A(x)$. Given $0 < y \leq T$, and some integer $n > 0$, let $\tau = y/n$ and approximate the tube from 0 to y as n uniform tubes of width τ, take the cross sectional area of tube k to be $A_k = A(\tau k)$. The reflection coefficient at the right boundary of the kth tube is therefore

$$r_k = \frac{A_{k+1} - A_k}{A_{k+1} + A_k}.$$

If s is any subset of the n tubes in this approximation, let $l(s)$ denote the sum of lengths of the sub-tubes in s, and $max(s)$ denote the right boundary of the last tube in s. Define

$$p(s, r_0, \ldots, r_n) = \text{the product of all } r_k\text{'s that are endpoints of a contiguous tube in } s.$$

and

$$F(x, y, \tau) = \sum_{x \leq l(s) < x+\tau} p(s, r_0, \ldots, r_n).$$

We now derive a recursive relationship for F and an integro-differential equation that its limit must satisfy. Partition the sum depending on the last sub-tube that is *not* in the system. For those subsystems where the nth tube is not in the system, we get a contribution

$$F(x, y - \tau, \tau).$$

For those subsystems that contain tubes $n, n-1, \ldots, n-j+1$ but not $n-j$, there is a factor $r_n r_{n-j}$ common to all $p(s, \ldots)$ and the remaining sum can be written

$$F(x - j\tau, y - (j+1)\tau, \tau).$$

Therefore we have

$$F(x, y, \tau) = F(x, y - \tau, \tau) + \sum_{j=1}^{\infty} r_n r_{n-j} F(x - j\tau, y - (j+1)\tau, \tau)$$

were we use the fact that $F(x, y, \tau) = 0$ if $x < 0$ or $y < 0$.

Let

$$r(y, \tau) = \frac{1}{\tau} \frac{A(y+\tau) - A(y)}{A(y+\tau) + A(y)}$$

so that

$$r_n = \tau\ r(y,\tau) \qquad r_{n-j} = \tau\ r(y - j\tau, \tau).$$

Note that as $\tau \to 0$, assuming A is sufficiently smooth, that

$$r(y,\tau) \to r(y) = \frac{A'(y)}{A(y)}.$$

Now introduce the notation for functions depending on variables written x and y,

$$\begin{array}{rcl} \Delta_y f(x,y,\tau) & = & \frac{f(x,y,\tau)-f(x,y-\tau,\tau)}{\tau} \\ \Sigma_{xy} f(x,y,\tau) & = & \sum_{j=1}^{\infty} f(x - j\tau, y - j\tau, \tau)\tau \end{array}$$

If as $\tau \to 0$, $f(x,y,\tau)$ converges with its derivatives in a well-behaved way, then we can conclude that

$$\begin{array}{rcl} \Delta_y f(x,y,\tau) & \to & \frac{\partial f(x,y)}{\partial y} \\ \Sigma_{xy} f(x,y,\tau) & \to & \int_{t=0}^{\infty} f(x-t, y-t)dt. \end{array}$$

as $\tau \to 0$. With this notation, the equation for F can be written

$$\Delta_y F(x,y,\tau) = \Sigma_{xy} r(y,\tau) F(x, y-\tau, \tau)$$

and so we conclude that if $F(x,y,\tau) \to F(x,y)$ as $\tau \to 0$, and convergence is well-behaved, then F must satisfy the integro-differential equation

$$\frac{\partial F(x,y)}{\partial y} = \int_{t=0}^{\infty} r(y-t) F(x-t, y-t)dt.$$

Now note that from the definition of $r(y)$ as the derivative of $\frac{1}{2}\log A(y)$, we can write the integral equivalently as a Riemann-Stieltjes integral

$$\frac{\partial F(x,y)}{\partial y} = \frac{1}{2}\int_{t=0}^{y} F(x-(y-t), t)d\log A(t).$$

These last two expressions are integro-differential equations representing the resonance of the composite tube. Since the system may equally well be represented by the wave equation, the integro-differential equations must be related to the wave equation.